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The ϕ_S polar decomposition of matrices

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ABSTRACT

Let $S \in M_{2n}$ be skew-symmetric and nonsingular. For $X \in M_{2n}$, we show that the following are equivalent: (a) X has a ϕ_S polar decomposition, (b) $\mathrm{rank}([X\phi_S(X)]^i) = \mathrm{rank}([\phi_S(X)X]^i)$ and $\mathrm{rank}([X\phi_S(X)]^i X)$ is even for all nonnegative integers i, and (c) $X\phi_S(X)$ is similar to $\phi_S(X)X$ and $\mathrm{rank}([X\phi_S(X)]^i X)$ is even for all nonnegative integer i.

1. Introduction

We let $M_{m,n}$ be the set of all m-by-n matrices with entries in $\mathbb{C}$, and we write $M_n \equiv M_{n,n}$. We let $\mathcal{S}_n$ be the set of all nonsingular skew-symmetric matrices in M_n (note that n is necessarily even). For $S \in \mathcal{S}_n$, we define

$$\phi_S : M_n \to M_n \text{ by } \phi_S(A) = S^{-1}A^TS \text{ for every } A \in M_n.$$

Let $S \in \mathcal{S}_{2n}$ and let $A \in M_{2n}$ be given. If $\phi_S(A) = A$, then A is said to be ϕ_S *symmetric*; while if $\phi_S(A) = A^{-1}$, then A is called ϕ_S *orthogonal*. We say that a matrix $A \in M_{2n}$ has a ϕ_S *polar decomposition* if there exist $Q, R \in M_{2n}$ such that Q is ϕ_S orthogonal, R is ϕ_S symmetric, and $A = QR$.

Every nonsingular $A \in M_{2n}$ has a ϕ_S polar decomposition; moreover, if (a not necessarily nonsingular) A has a ϕ_S polar decomposition then $A\phi_S(A)$ is similar to $\phi_S(A)A$ and $\mathrm{rank}(A)$ is even [3]. If A is ϕ_S symmetric, then $A^TS = SA = -(A^TS)^T$ is skew-symmetric; and hence, A is a product of two skew-symmetric matrices (one of which is nonsingular) and has even rank [6, Lemma 4.3]. Moreover, if A

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Table 1
Symplectic operations.

Symplectic operation	Column operation	Row operation
TYPE $A_I(a,i)$	$F_i \to aF_i$	$R_i \to aR_i$
$i \in \{1,\ldots,n\}$	$F_{i+n} \to a^{-1}F_{i+n}$	$R_{i+n} \to a^{-1}R_{i+n}$
TYPE $A_{II}(a,i,j)$	$F_j \to aF_i + F_j$	$R_j \to aR_i + R_j$
$i,j \in \{1,\ldots,n\}$	$F_{i+n} \to -aF_{j+n} + F_{i+n}$	$R_{i+n} \to -aR_{j+n} + R_{i+n}$
TYPE $A_{III}(i,j)$	$F_i \leftrightarrow F_j$	$R_i \leftrightarrow R_j$
$i,j \in \{1,\ldots,n\}$	$F_{i+n} \leftrightarrow F_{j+n}$	$R_{i+n} \leftrightarrow R_{j+n}$
TYPE $B(i)$	$F_i \leftrightarrow F_{i+n}$	$R_i \leftrightarrow R_{i+n}$
$i \in \{1,\ldots,n\}$	then $F_i \to -F_i$	then $R_{i+n} \to -R_{i+n}$
TYPE $C_I(a,i,j)$	$F_i \to aF_j + F_i$	$R_j \to aR_i + R_j$
$i \in \{n+1,\ldots,2n\}, j \in \{1,\ldots,n\}$	$F_{j+n} \to aF_{i-n} + F_{j+n}$	$R_{i-n} \to aR_{j+n} + R_{i-n}$
TYPE$C_{II}(b,k,l)$	$F_k \to bF_l + F_k$	$R_l \to bR_k + R_l$
$k \in \{1,\ldots,n\}, l \in \{n+1,\ldots,2n\}$	$F_{l-n} \to bF_{k+n} + F_{l-n}$	$R_{k+n} \to bR_{l-n} + R_{k+n}$

is ϕ_S symmetric then A^k is also ϕ_S symmetric for every nonnegative integer k. If $A = QR$ is a ϕ_S polar decomposition of A, then $[A\phi_S(A)]^k A = QR^{2k+1}$ so that $\operatorname{rank}([A\phi_S(A)]^k A)$ is even for all k; similarly, $\operatorname{rank}([\phi_S(A)A]^k \phi_S(A))$ is even for all k.

Set $J_{2n} \equiv \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \in M_{2n}$. If it is clear from the context, we simply denote J_{2n} by J. If $A, B, C, D \in M_n$ and

$$\text{if } M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ then } \phi_J(M) = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}. \tag{1}$$

A ϕ_J orthogonal matrix is also called *symplectic* while a ϕ_J symmetric matrix is also called *skew-Hamiltonian*.

For $A, B \in M_{2n}$, we say that A is *symplectically equivalent* to B if $B = PAQ$ for some symplectic P and Q. If A is symplectically equivalent to B, then A has a ϕ_J polar decomposition if and only if B has a ϕ_J polar decomposition [1].

Let $A_i \in M_m$ and $B_i \in M_n$ for each $i = 1, 2, 3, 4$; let $A \equiv \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in M_{2m}$ and let $B \equiv \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in M_{2n}$. The *expanding sum* of A and B is given by

$$A \boxplus B = \begin{bmatrix} A_1 \oplus B_1 & A_2 \oplus B_2 \\ A_3 \oplus B_3 & A_4 \oplus B_4 \end{bmatrix}. \tag{2}$$

If each of A and B has a ϕ_J polar decomposition, then so does the expanding sum of A and B [5, Theorem 3].

In [1,5] symplectic matrix operations are used to determine whether a matrix of rank at most 4 has a ϕ_J polar decomposition. When multiplied to the left of a matrix A, the symplectic matrix operations affect the rows of A and we call them *symplectic row operations*. When multiplied to the right, the symplectic matrix operations affect the columns of the matrix and are called *symplectic column operations*.

The following table shows the effect of symplectic operations on a matrix [5]. Here, the columns are represented by F_i, while the rows are represented by R_i. For $a \in \mathbb{C}$, the notation $F_i \to aF_j + F_i$ means that to obtain the new column F_i, we add aF_j to the old F_i; the notation $F_i \leftrightarrow F_j$ means that we switch columns i and j. Similar notations are used for R_i.

After performing symplectic operations on matrix A to obtain a new matrix A_1, we continually refer to the rows of A_1 as R_i and to the columns of A_1 as F_i.

Let n be a positive integer. Set $N_1 \equiv \{1,\ldots,n\}$, set $N_2 \equiv \{n+1,\ldots,2n\}$, and set $N_3 \equiv N_1 \cup N_2$. Let i and j be integers with $0 \leqslant i, j \leqslant n$. If $i = 0$, then we set $\alpha_0 = \beta_0 = \varnothing$. Otherwise, we set $\alpha_i \equiv$

$\{1, \ldots, i\}$, we set $\beta_i \equiv \{n+1, \ldots, n+i\}$, and we set $\delta_{ij} \equiv \alpha_i \cup \beta_j$. Moreover, we set $\alpha_i^C \equiv N_1 - \alpha_i$, we set $\beta_i^C \equiv N_2 - \beta_i$, and we set $\delta_{ij}^C \equiv N_3 - \delta_{ij}$.

Let $A \in M_n$ be given. For $\alpha, \beta \subset N_1$, the submatrix of A containing the rows indexed by α and the columns indexed by β is $A(\alpha, \beta)$ [2]. We also use the convention that matrices of size 0 are empty and that $A^0 \equiv I$ for any matrix $A \in M_n$. For $A \in M_{2n}$, we say that A has the form $R(i,j)F(k,l)$ if $R_t = 0$ for every $t \in \delta_{ij}^C$ and if $F_u = 0$ for every $u \in \delta_{kl}^C$.

Let $A \in M_{2n}$ with $\operatorname{rank}(A) = 2k$ be given. Then, Corollary 11 and Theorem 12 of [5] guarantee that there exist integers i, j with $k \leqslant i, j \leqslant \min\{n, 2k\}$ such that

1. for some symplectic $Q, P \in M_{2n}$, we have QAP is of the form $R(i, 2k-i)F(j, 2k-j)$;
2. $\operatorname{rank}(A\phi_J(A)) = 2(2k-j)$; and
3. $\operatorname{rank}(\phi_J(A)A) = 2(2k-i)$.

If $A \in M_{2n}$ has a ϕ_J polar decomposition, then $A\phi_J(A)$ is similar to $\phi_J(A)A$; it follows that if $i \neq j$, then A does not have a ϕ_J polar decomposition. Hence, to determine those $A \in M_{2n}$ with ϕ_J polar decomposition, we only need to look at matrices with even ranks and which are symplectically equivalent to matrices of the form $R(i, 2k-i)F(i, 2k-i)$. We study such matrices and determine which ones have ϕ_J polar decompositions.

2. The form $R(i,j)F(i,j)$

Let $M \in M_{2n}$ be given, suppose that $i \geqslant j$, and suppose that M has the form $R(i,j)F(i,j)$. Then there exist blocks $A, C, G, K \in M_j$, blocks $B, H, D^T, F^T \in M_{j,i-j}$, and $E \in M_{i-j}$ such that M can be partitioned as

$$M = \begin{bmatrix} A & B & 0 & C & 0 & 0 \\ D & E & 0 & F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ G & H & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Using such a partition for M, the following can be easily shown.

Lemma 1. *Let i, j, n be given integers. Suppose that n is positive and that $0 \leqslant j \leqslant i \leqslant n$. Let $A, B \in M_{2n}$ be given.*

1. *If A has the form $R(i,j)F(i,j)$, then $\phi_J(A)$ has the form $R(j,i)F(j,i)$.*
2. *If A and B have the form $R(i,j)F(i,j)$, then so does AB. Moreover, $AB(\delta_{ij}, \delta_{ij}) = A(\delta_{ij}, \delta_{ij})B(\delta_{ij}, \delta_{ij})$.*
3. *If A and B have the form $R(j,i)F(j,i)$, then so does AB. Moreover, $AB(\delta_{ji}, \delta_{ji}) = A(\delta_{ji}, \delta_{ji})B(\delta_{ji}, \delta_{ji})$.*
4. *If A and B have the form $R(i,j)F(j,i)$, then so does AB. Moreover,*

   (a) $AB(\delta_{ij}, \delta_{ji}) = A(\delta_{ij}, \delta_{jj})B(\delta_{jj}, \delta_{ji})$,

   (b) $AB(\delta_{ij}, \delta_{jj}) = A(\delta_{ij}, \delta_{jj})B(\delta_{jj}, \delta_{jj})$, and

   (c) $AB(\delta_{jj}, \delta_{ji}) = A(\delta_{jj}, \delta_{jj})B(\delta_{jj}, \delta_{ji})$.

5. *If A and B have the form $R(j,i)F(i,j)$, then so does AB. Moreover,*

   (a) $AB(\delta_{ji}, \delta_{ij}) = A(\delta_{ji}, \delta_{jj})B(\delta_{jj}, \delta_{ij})$,

   (b) $AB(\delta_{ji}, \delta_{jj}) = A(\delta_{ji}, \delta_{jj})B(\delta_{jj}, \delta_{jj})$, and

   (c) $AB(\delta_{jj}, \delta_{ij}) = A(\delta_{jj}, \delta_{jj})B(\delta_{jj}, \delta_{ij})$.

Lemma 2. *Let i, j, n be given integers. Suppose that n is positive and that $0 \leqslant j \leqslant i \leqslant n$. Let $A, B \in M_{2n}$ be given.*

1. *Suppose A has the form $R(i,j)F(i,j)$ and B has the form $R(j,i)F(i,j)$. Then*
   (a) *AB has the form $R(i,j)F(i,j)$,*
   (b) *$AB(\delta_{ij}, \delta_{ij}) = A(\delta_{ij}, \delta_{jj})B(\delta_{jj}, \delta_{ij})$,*
   (c) *BA has the form $R(j,i)F(i,j)$, and*
   (d) *$BA(\delta_{ji}, \delta_{ij}) = B(\delta_{ji}, \delta_{ij})A(\delta_{ij}, \delta_{ij})$.*
2. *Suppose A has the form $R(i,j)F(i,j)$ and B has the form $R(j,i)F(j,i)$. Then*
   (a) *AB has the form $R(i,j)F(j,i)$,*
   (b) *$AB(\delta_{ij}, \delta_{ji}) = A(\delta_{ij}, \delta_{jj})B(\delta_{jj}, \delta_{ji})$,*
   (c) *BA has the form $R(j,i)F(i,j)$, and*
   (d) *$BA(\delta_{ji}, \delta_{ij}) = B(\delta_{ji}, \delta_{jj})A(\delta_{jj}, \delta_{ij})$.*
3. *Suppose A has the form $R(i,j)F(i,j)$ and B has the form $R(i,j)F(j,i)$. Then*
   (a) *AB has the form $R(i,j)F(j,i)$,*
   (b) *$AB(\delta_{ij}, \delta_{ji}) = A(\delta_{ij}, \delta_{ij})B(\delta_{ij}, \delta_{ji})$,*
   (c) *BA has the form $R(i,j)F(i,j)$, and*
   (d) *$BA(\delta_{ij}, \delta_{ij}) = B(\delta_{ij}, \delta_{jj})A(\delta_{jj}, \delta_{ij})$.*
4. *Suppose A has the form $R(j,i)F(i,j)$ and B has the form $R(j,i)F(j,i)$. Then*
   (a) *AB has the form $R(j,i)F(j,i)$,*
   (b) *$AB(\delta_{ji}, \delta_{ji}) = A(\delta_{ji}, \delta_{jj})B(\delta_{jj}, \delta_{ji})$,*
   (c) *BA has the form $R(j,i)F(i,j)$, and*
   (d) *$BA(\delta_{ji}, \delta_{ij}) = B(\delta_{ji}, \delta_{ji})A(\delta_{ji}, \delta_{ij})$.*

Let $M \in M_{2n}$ and let integers i, j be given with $0 < j \leqslant i \leqslant n$. Suppose that M is of the form $R(i,j)F(i,j)$. Partition M as in (3), and set $\mathcal{B} \equiv M\phi_J(M)$. Using (1), we have

$$\phi_J(M) = \begin{bmatrix} K^T & 0 & 0 & -C^T & -F^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -G^T & 0 & 0 & A^T & D^T & 0 \\ -H^T & 0 & 0 & B^T & E^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

and a direct calculation shows that

$$\mathcal{B} = \begin{bmatrix} AK^T - CG^T & 0 & 0 & -AC^T + CA^T & -AF^T + CD^T & 0 \\ DK^T - FG^T & 0 & 0 & -DC^T + FA^T & -DF^T + FD^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ GK^T - KG^T & 0 & 0 & -GC^T + KA^T & -GF^T + KD^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

Set

$$Y_1 \equiv \begin{bmatrix} A & C \\ D & F \\ G & K \end{bmatrix}, \; Y_2 \equiv \begin{bmatrix} K^T & -C^T & -F^T \\ -G^T & A^T & D^T \end{bmatrix}, \text{ and } Y_3 \equiv \begin{bmatrix} A & B & C \\ G & H & K \end{bmatrix}. \tag{6}$$

Notice that $Y_1 = M(\delta_{ij}, \delta_{jj})$, that $Y_2 = \phi_J(M)(\delta_{jj}, \delta_{ji})$, and that $Y_3 = M(\delta_{jj}, \delta_{ij})$.

Definition 3. Let $M \in M_{2n}$ have the form $R(i,j)F(i,j)$ and be partitioned as in (3). We define

$$\mathcal{Y}(M) \equiv M\left(\delta_{jj}, \delta_{jj}\right) = \begin{bmatrix} A & C \\ G & K \end{bmatrix} \in M_{2j}. \tag{7}$$

Lemma 2 (2b) guarantees that $\mathcal{B}(\delta_{ij}, \delta_{ji}) = M(\delta_{ij}, \delta_{jj})\phi_J(M)(\delta_{jj}, \delta_{ji}) = Y_1 Y_2$. Moreover, one checks that

$$\mathcal{B}\left(\delta_{ij}, \delta_{jj}\right) = Y_1 \phi_J\left(\mathcal{Y}(M)\right), \tag{8}$$

that

$$\mathcal{B}\left(\delta_{jj}, \delta_{ji}\right) = \mathcal{Y}(M)\, Y_2, \tag{9}$$

and that

$$\mathcal{B}\left(\delta_{jj}, \delta_{jj}\right) = \mathcal{Y}(M)\, \phi_J\left(\mathcal{Y}(M)\right). \tag{10}$$

Set $\mathcal{C} \equiv \phi_J(\mathcal{Y}(M))\mathcal{Y}(M)$ and notice that Lemma 1 (4a) guarantees that

$$\mathcal{B}^2\left(\delta_{ij}, \delta_{ji}\right) = \mathcal{B}\left(\delta_{ij}, \delta_{jj}\right) \mathcal{B}\left(\delta_{jj}, \delta_{ji}\right) = Y_1 \mathcal{C} Y_2.$$

We also have

$$\mathcal{B}^2\left(\delta_{ij}, \delta_{jj}\right) = \mathcal{B}\left(\delta_{ij}, \delta_{jj}\right) \mathcal{B}\left(\delta_{jj}, \delta_{jj}\right) = Y_1 \mathcal{C} \phi_J\left(\mathcal{Y}(M)\right). \tag{11}$$

Now, $\mathcal{B}^p(\delta_{ij}, \delta_{ji}) = \mathcal{B}^{p-1}(\delta_{ij}, \delta_{jj})\mathcal{B}(\delta_{jj}, \delta_{ji})$, so that Eqs. (9) and (11) and a repeated use of Lemma 1 (4b) show that for every positive integer p,

$$\mathcal{B}^p\left(\delta_{ij}, \delta_{ji}\right) = Y_1 \mathcal{C}^{p-1} Y_2. \tag{12}$$

Moreover, we have $\mathcal{B}^p M(\delta_{ij}, \delta_{ij}) = \mathcal{B}^p(\delta_{ij}, \delta_{jj})M(\delta_{jj}, \delta_{ij})$, so that for every positive integer p,

$$\mathcal{B}^p M\left(\delta_{ij}, \delta_{ij}\right) = Y_1 \mathcal{C}^{p-1} \phi_J\left(\mathcal{Y}(M)\right) Y_3. \tag{13}$$

Suppose now that M has rank $2k \equiv i + j$. Then each of Y_1, Y_2 , and Y_3 in (6) have full rank $2j$. Hence, for every positive integer p, we have $\operatorname{rank}(\mathcal{B}^p) = \operatorname{rank}(\mathcal{B}^p(\delta_{ij}, \delta_{ji})) = \operatorname{rank}(\mathcal{C}^{p-1})$. Moreover, we also have $\operatorname{rank}(\mathcal{B}^p M) = \operatorname{rank}(\mathcal{B}^p M(\delta_{ij}, \delta_{ij})) = \operatorname{rank}(\mathcal{C}^{p-1}\phi_J(\mathcal{Y}(M)))$.

Set $Y_4 \equiv \phi_J(M)(\delta_{ji}, \delta_{jj}) = \begin{bmatrix} K^T & -C^T \\ -G^T & A^T \\ -H^T & B^T \end{bmatrix}$, set $\mathcal{D} \equiv \phi_J(M)M$, and set $\mathcal{E} = \mathcal{Y}(M)\phi_J(\mathcal{Y}(M))$.

One checks that

$$\mathcal{D}\left(\delta_{ji}, \delta_{jj}\right) = Y_4 \mathcal{Y}(M), \tag{14}$$

that

$$\mathcal{D}\left(\delta_{jj}, \delta_{ij}\right) = \phi_J\left(\mathcal{Y}(M)\right) Y_3, \tag{15}$$

and that

$$\mathcal{D}\left(\delta_{jj}, \delta_{jj}\right) = \phi_J\left(\mathcal{Y}(M)\right) \mathcal{Y}(M) = \mathcal{C}. \tag{16}$$

Now, Lemma 1 (5a) and Eqs. (14) and (15) guarantee that

$$\mathcal{D}^2\left(\delta_{ji}, \delta_{ij}\right) = \mathcal{D}\left(\delta_{ji}, \delta_{jj}\right) \mathcal{D}\left(\delta_{jj}, \delta_{ij}\right) = Y_4 \mathcal{E} Y_3.$$

We also have

$$\mathcal{D}^2\left(\delta_{ji}, \delta_{jj}\right) = \mathcal{D}\left(\delta_{ji}, \delta_{jj}\right) \mathcal{D}\left(\delta_{jj}, \delta_{jj}\right) = Y_4 \mathcal{Y}(M)\, \mathcal{C} = Y_4 \mathcal{E} Y(M). \tag{17}$$

Notice that $\mathcal{D}^p(\delta_{ji}, \delta_{ij}) = \mathcal{D}^{p-1}(\delta_{ji}, \delta_{jj})\mathcal{D}(\delta_{jj}, \delta_{ij})$, so that Eqs. (15) and (16) and a repeated use of Lemma 1 (5b) show that for every positive integer p,

$$\mathcal{D}^p\left(\delta_{ji}, \delta_{ij}\right) = Y_4 \mathcal{E}^{p-1} Y_3. \tag{18}$$

Moreover, we have $\mathcal{D}^p \phi_J(M)(\delta_{ji}, \delta_{ji}) = \mathcal{D}^p(\delta_{ji}, \delta_{jj})\phi_J(M)(\delta_{jj}, \delta_{ji})$, so that for every positive integer p,

$$\mathcal{D}^p \phi_J\left(M\right)\left(\delta_{ji}, \delta_{ij}\right) = Y_4 \mathcal{E}^{p-1} \mathcal{Y}\left(M\right) Y_2. \tag{19}$$

If M has rank $2k \equiv i + j$, then Y_4 (just like Y_1, Y_2, and Y_3) also has full rank.

Theorem 4. *Let $M \in M_{2n}$ be given. If $\operatorname{rank}(M) = 2k > 0$ and if M is of the form $R(i,j)F(i,j)$, where $0 \leqslant j = 2k - i \leqslant i \leqslant n$, then*

$$\operatorname{rank}([M\phi_J(M)]^p M^l) = \operatorname{rank}([\phi_J(\mathcal{Y}(M))\mathcal{Y}(M)]^{p-1}\left[\phi_J\left(\mathcal{Y}(M)\right)\right]^l) \tag{20}$$

and

$$\operatorname{rank}([\phi_J(M)M]^p\left(\phi_J\left(M\right)\right)^l) = \operatorname{rank}([\mathcal{Y}(M)\phi_J(\mathcal{Y}(M))]^{p-1}\left[\mathcal{Y}(M)\right]^l), \tag{21}$$

for every positive integer p and every $l \in \{0, 1\}$.

Let $A \in M_{2n}$ have rank $2k$. If $\operatorname{rank}(A\phi_J(A)) = \operatorname{rank}(\phi_J(A)A)$, then Corollary 11 and Theorem 12 of [5] guarantee that there exist symplectic $P, Q \in M_{2n}$, integers $i \geqslant j = 2k - i \geqslant 0$, and $X \in M_{2n}$ having the form $R(i,j)F(i,j)$ such that $A = PXQ$. We define $\mathcal{Y}_{P,Q}(A) \equiv \mathcal{Y}(X)$.

Notice that if $A = PXQ$ is any such factorization of A, then Eqs. 20 and 21 of Theorem 4 are satisfied when we replace M (on the left hand side) by A and if we replace $\mathcal{Y}(M)$ by $\mathcal{Y}_{P,Q}(A)$. For this reason, we use $\mathcal{Y}(A)$ instead of $\mathcal{Y}_{P,Q}(A)$. Moreover, we write $\mathcal{Y}^0(A) = A$ and whenever defined, we write $\mathcal{Y}^l(A) = \mathcal{Y}(\mathcal{Y}^{l-1}(A))$.

Theorem 4 guarantees that $\operatorname{rank}(A\phi_J(A)A) = \operatorname{rank}(\phi_J(\mathcal{Y}(A)))$. Suppose further that $\operatorname{rank}(A\phi_J(A)A)$ is even and that $\operatorname{rank}([A\phi_J(A)]^2) = \operatorname{rank}([\phi_J(A)A]^2)$. Then, we must also have $\operatorname{rank}(\mathcal{Y}(A))$ is even (say, $2p$), $\operatorname{rank}(\mathcal{Y}(A)\phi_J(\mathcal{Y}(A))) = \operatorname{rank}(\phi_J(\mathcal{Y}(A))\mathcal{Y}(A))$, and there exist symplectic P and Q, and integers $l \geqslant m = 2p - l \geqslant 0$ so that $P\mathcal{Y}(A)Q$ has the form $R(l,m)F(l,m)$. Thus, $\mathcal{Y}^2(A) = \mathcal{Y}_{P,Q}(\mathcal{Y}(A))$ exists.

Similarly, if i is a positive integer and if for every $l \in \alpha_i = \{1, \ldots, i\}$ we have $\operatorname{rank}([A\phi_J(A)]^{l-1}A)$ is even and $\operatorname{rank}([A\phi_J(A)]^l) = \operatorname{rank}([\phi_J(A)A]^l)$, then $\mathcal{Y}^l(A)$ exists for each $l \in \alpha_i$.

3. The ϕ_J polar decomposition

We begin with the following observation. Let $A, P, Q \in M_{2n}$ be given with P and Q symplectic.If $X \equiv PAQ$, then $X\phi_J(X)$ is similar to $A\phi_J(A)$ and $\phi_J(X)X$ is similar to $\phi_J(A)A$.

Lemma 5. *Let $M, P, Q \in M_{2n}$ be given with P and Q symplectic. Let $X = PMQ$. Then*

1. $\operatorname{rank}(M\phi_J(M)) = \operatorname{rank}(X\phi_J(X))$,
2. $\operatorname{rank}(\phi_J(M)M) = \operatorname{rank}(\phi_J(X)X)$, *and*
3. $\operatorname{rank}([M\phi_J(M)]^l M) = \operatorname{rank}([X\phi_J(X)]^l X)$ *for every integer $l \geqslant 0$.*

Let $A \in M_{2n}$ be given. Then, $\operatorname{rank}(A) = \operatorname{rank}(\phi_J(A))$. Hence, in Lemma 5, we also have $\operatorname{rank}([\phi_J(M)M]^l \phi_J(M)) = \operatorname{rank}([\phi_J(X)X]^l \phi_J(X))$ for every integer $l \geqslant 0$.

Let $B \in M_{2n}$ have rank $2k$ and be of the form $R(i,j)F(i,j)$, with $j = 2k - i$. If $j = 0$, then Corollary 11 of [5] guarantees that $2k = i \leqslant n$. Thus, $B = C \oplus 0$, with $C \in M_i$ nonsingular. Theorem 4 of [5] now ensures that B has a ϕ_J polar decomposition. Therefore, we only look at the case $j > 0$.

Lemma 6. *Let $X \in M_{2n}$ be given with $\operatorname{rank}(X) = 2k > 0$. Suppose that X is of the form $R(i,j)F(i,j)$, with $i \geqslant j = 2k - i > 0$, and suppose further that $\mathcal{Y}(X)\phi_J(\mathcal{Y}(X))$ is nilpotent. If $\mathcal{Y}(X)$ has a ϕ_J polar decomposition then X has a ϕ_J polar decomposition.*

Proof. Let $X \in M_{2n}$ and $\operatorname{rank}(X) = 2k > 0$. Suppose that X is of the form $R(i,j)F(i,j)$ with $i \geqslant j > 0$, suppose that $\mathcal{Y}(X)\phi_J(\mathcal{Y}(X))$ is nilpotent and suppose that $\mathcal{Y}(X)$ has a ϕ_J polar decomposition, say

$\mathcal{Y}(X) = LR$ where L is ϕ_J orthogonal and R is ϕ_J symmetric. Then, $LR^2L^{-1} = \mathcal{Y}(X)\phi_J(\mathcal{Y}(X))$ is nilpotent, so that R^2 and hence R, is nilpotent.

Using Proposition 5 of [5] (see also [4]), $\mathcal{Y}(X)$ is symplectically equivalent to (or R is symplectically similar to) $N \oplus N^T$, where N is a direct sum of Jordan blocks corresponding to 0. Suppose $P[\mathcal{Y}(X)]Q = N \oplus N^T$, where P and Q are both symplectic. Then $P \boxplus I_{2(n-j)}$ and $Q \boxplus I_{2(n-j)}$ are both symplectic and

$$X_1 \equiv (P \boxplus I)X(Q \boxplus I) = \begin{bmatrix} N & B & 0 & 0 & 0 & 0 \\ D & E & 0 & F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & H & 0 & N^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has the form (3) with $C = 0$, $G = 0$, and $K = N^T$. Set

$$X_2 \equiv \begin{bmatrix} N & B & 0 & 0 \\ D & E & F & 0 \\ 0 & H & N^T & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_{2i},$$

and notice that $X_1 = X_2 \boxplus 0_{2n-2i}$. We apply symplectic column and row operations on X_2 to reduce it to an expanding sum of two matrices, each of which has a ϕ_J polar decomposition.

First, observe that N has a nonzero entry on row (column) p if and only if N^T has a nonzero entry on column (row) p. Now, for every nonzero row p of N we use a type A_{II} symplectic column operation to zero out row p of B. Note that this only affects E. Also, for every nonzero column p of N^T we use a type C_{II} symplectic row operation to zero out column p of F. Again, this only affects E.

We apply the same reduction on the transpose of (the new) X_2 to get (recall that after applying symplectic operations, we continually refer to row i as R_i; to column i as F_i; and similarly for matrices B , D, E, F, and H)

$$X_3 = \begin{bmatrix} N & B & 0 & 0 \\ D & E & F & 0 \\ 0 & H & N^T & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with the following properties:

(**a**) row p of B is zero if and only if row p of N is nonzero if and only if column p of N^T is nonzero if and only if column p of F is zero;
(**b**) row q of H is zero if and only if row q of N^T is nonzero if and only if column q of N is nonzero if and only if column q of D is zero.

For a given positive integer n, we let e_i be the ith standard basis vector of $\mathbb{C}^n$, that is, the ith entry of e_i is 1 and all other entries are 0.

Notice that we can apply a combination of type A_I, A_{II}, A_{III} symplectic column operations to X_3 to make the first nonzero row of B equal to e_1^T. Doing so only affects E, but retains properties (**a**) and (**b**). Also, the second (updated) nonzero row of B cannot be linearly dependent on the first nonzero row of B, which is now e_1^T. Hence, it must have a nonzero entry other than the first column entry and so we can apply a combination of types A_I, A_{II}, A_{III} symplectic column operations on X_3 to make the second nonzero row of B equal to e_2^T. Doing such a procedure preserves properties (**a**) and (**b**). Apply the same process to make every nonzero row r of B equal to e_r^T and notice that properties (**a**) and (**b**) are retained.

Let s be the number of Jordan blocks of N. Note that B and H each have s nonzero rows and together the nonzero rows of B and H are linearly independent since they correspond to the zero rows of N and N^T. The first nonzero row of H must be linearly independent of $\{e_1^T, e_2^T, \ldots, e_s^T\}$, which are the nonzero rows of B. Hence we can apply a combination of types A_I, A_{II}, A_{III} symplectic column operations on X_3 to make the first nonzero row of H equal to e_{s+1}^T. Again, such a process preserves properties (**a**) and (**b**). Since the number of zero rows of N is equal to the number of zero rows of N^T, we can apply the same type of procedure to make the nonzero row r of H equal to e_{s+r}^T and here the last nonzero row of H is equal to e_{2s}^T.

Apply a combination of types A_I, A_{II}, A_{III} symplectic row operations on X_3 to make the first nonzero column of F equal to e_1. This preserves properties (**a**) and (**b**) but does not affect B and H. Note that the first nonzero row of B is equal to the transpose of the first nonzero column of F. Now, the second nonzero column of F cannot be linearly dependent on the first nonzero column of F , which is now e_1. Hence, we can apply a combination of types A_I, A_{II}, A_{III} symplectic row operations on X_3 to make the second nonzero column of F equal to e_2. Use a similar procedure to make every nonzero column r of F equal to e_r and notice that $B = F^T$.

The nonzero columns of D and F together form a linearly independent set since they correspond to the zero columns of N and N^T. The first nonzero column of D is linearly independent of $\{e_1, e_2, \ldots, e_s\}$, which are the nonzero columns of F. We can then apply a combination of types A_I, A_{II}, A_{III} symplectic row operations to make the first nonzero column of D equal to e_{s+1} and which leaves B , H and F unaffected. Notice that the first nonzero column of D is equal to the transpose of the first nonzero row of H. The second nonzero column of D and e_{s+1} cannot be linearly independent, and hence we can apply a combination of types A_I, A_{II}, A_{III} symplectic row operations to make the second nonzero column of D equal to e_{s+2}. Using a similar process, we can make the nonzero row r of D equal to e_{s+r} and note that $D = H^T$.

Notice that properties (**a**) and (**b**) are preserved, and X_3 has now been symplectically reduced to

$$X_3 = \begin{bmatrix} N & B_1 & 0 & 0 & 0 & 0 \\ 0 & E_1 & E_2 & E_3 & B_1^T & 0 \\ D_1 & E_4 & E_5 & E_6 & 0 & 0 \\ 0 & E_7 & E_8 & E_9 & 0 & 0 \\ 0 & 0 & D_1^T & 0 & N^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $B_1, D_1^T \in M_{j,s}$ and $E \equiv \begin{bmatrix} E_1 & E_2 & E_3 \\ E_4 & E_5 & E_6 \\ E_7 & E_8 & E_9 \end{bmatrix} \in M_{i-j}$.

Zero out the first s columns of E by type A_{II} symplectic row operations using the s nonzero rows of B_1. Property (**a**) assures that doing so does not affect any other entries of X_2. Zero out the next s columns of E by type C_{II} symplectic row operations using the s nonzero columns of D_1^T and, by property (**b**), nothing else is affected by this reduction.

Apply similar arguments to X_3^T to symplectically reduce X_3 to

$$X_5 \boxplus (E_9 \oplus 0_{i-j-2s}),$$

where

$$X_5 \equiv \begin{bmatrix} N & B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1^T & 0 & 0 \\ D_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_1^T & N^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that $\operatorname{rank}(X_5) = 2j + 2s$, and since $\operatorname{rank}(X_2) = 2k$, we have $\operatorname{rank}(E_9) = i - j - 2s = 2k - 2j - 2s$. Hence, E_9 is nonsingular. One checks that $X_5 = QS$, where $Q \equiv P \boxplus T$, $P \equiv I_j \oplus I_j$,

$$T \equiv \begin{bmatrix} 0 & 0 & I_{i-j} & 0 \\ 0 & 0 & 0 & -I_{n-i} \\ -I_{i-j} & 0 & 0 & 0 \\ 0 & I_{n-i} & 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} N & B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_1^T & N^T & 0 & 0 \\ 0 & 0 & 0 & B_1^T & 0 & 0 \\ -D_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

moreover, Q is ϕ_J orthogonal and S is ϕ_J symmetric.

Finally, since X is symplectically equivalent to $X_5 \boxplus (E_9 \oplus 0) \boxplus 0$, we conclude that X has a ϕ_J polar decomposition. $\square$

Let $M \in M_{2n}$ have rank $2k$ and have the form (3). We now give a sufficient condition for M to have a ϕ_J polar decomposition.

Theorem 7. *Let $M \in M_{2n}$ with $\operatorname{rank}(M) = 2k > 0$. Suppose further that M is of the form $R(i,j)F(i,j)$ with $i \geqslant j = 2k - i > 0$. If $\mathcal{Y}(M)$ has a ϕ_J polar decomposition, then M has a ϕ_J polar decomposition.*

Proof. Let $M \in M_{2n}$ with $\operatorname{rank}(M) = 2k > 0$. Let M be of the form $R(i,j)F(i,j)$ with $i \geqslant j = 2k - i > 0$. Suppose $\mathcal{Y}(M)$ has a ϕ_J polar decomposition. Then $\mathcal{Y}(M)$ is symplectically equivalent to $N_1 \oplus N_2 \oplus N_1^T \oplus N_2^T$ where N_1 is nonsingular and N_2 is nilpotent [4, Theorem 3] (see also [5, Proposition 5]), that is, there exist symplectic $P, Q \in M_{2j}$ such that $P[\mathcal{Y}(M)]Q = N_1 \oplus N_2 \oplus N_1^T \oplus N_2^T$. Now, $P_1 \equiv P \boxplus I_{2(n-j)}$ and $Q \equiv Q \boxplus I_{2(n-j)}$ are both symplectic and

$$X_1 \equiv P_1 M Q_1 = \begin{bmatrix} N_1 \oplus N_2 & B & 0 & 0 & 0 \\ D & E & 0 & F & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & H & 0 & N_1^T \oplus N_2^T & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $B, H, D^T, F^T \in M_{j,i-j}$, $E \in M_{i-j}$.

Suppose $N_1 \in M_m$. Since N_1 is nonsingular, we can zero out the first m rows of B using type A_{II} symplectic column operations. Columns $n+1$ up to $2n$ are unchanged. Similarly, N_1^T is nonsingular, so we can use type C_{II} symplectic row operations to zero out the first m columns of F. This only affects E and F. Applying the same reduction to (the new) X_1^T symplectically reduces M to $(N_1 \oplus N_1^T) \boxplus M_2$; where M_2 satisfies the conditions of Lemma 6 and hence, has a ϕ_J polar decomposition. Since $N_1 \oplus N_1^T$ is ϕ_J symmetric, then $M = (N_1 \oplus N_1^T) \boxplus M_2$ has a ϕ_J polar decomposition. $\square$

Theorem 8. *Let $M \in M_{2n}$ with rank $(M) = 2k > 0$. Suppose that for every nonnegative integer t, we have (i) $\operatorname{rank}(M\phi_J(M))^t = \operatorname{rank}(\phi_J(M)M)^t$ and (ii) rank $(M\phi_J(M))^t M$ is even. Then M has a ϕ_J polar decomposition.*

Proof. Under the assumed conditions, $\mathcal{Y}^t(M)$ is defined for every nonnegative integer t.

Suppose that $M\phi_J(M)$ is nilpotent. Then there exists a positive integer t_o such that $(M\phi_J(M))^{t_o}M = 0$ but $(M\phi_J(M))^{t_o-1}M \neq 0$. In this case, $\mathcal{Y}^{t_o-1}(M)$ exists and is of even rank and $\mathcal{Y}(\mathcal{Y}^{t_o-1}(M)) = 0$. By Lemma 6, $\mathcal{Y}^{t_o-1}(M)$ has a ϕ_J polar decomposition. Applying Theorem 7 $t_o - 1$ times, we conclude that $\mathcal{Y}^t(M)$ has a ϕ_J polar decomposition, for all $0 < t < t_o - 1$, hence M has a ϕ_J polar decomposition.

Suppose that $M\phi_J(M)$ is not nilpotent, so that $\operatorname{rank}(M\phi_J(M))^t > 0$ for every integer $t \geqslant 0$. Notice that $\operatorname{rank}(\mathcal{Y}^l(M)) \geqslant \operatorname{rank}(\mathcal{Y}^{l+1}(M))$for every integer $l \geqslant 0$. Let t_0 be the least positive integer so that $\operatorname{rank}(\mathcal{Y}^{t_0}(M)) = \operatorname{rank}(\mathcal{Y}^{t_0+1}(M))$. Then $\mathcal{Y}^{t_0}(M)$ is nonsingular and hence has a ϕ_J polar decomposition. Applying Theorem 7 t_0 times we conclude that $\mathcal{Y}^t(M)$ has a ϕ_J polar decomposition, for all $0 < t < t_0 - 1$, and thus, M has a ϕ_J polar decomposition. $\square$

We note that although conditions (i) and (ii) of Theorem 8 require checking infinite values of t, the same conclusion can be made by checking conditions (i) and (ii) for $0 \leqslant t \leqslant n$.

The next result follows immediately from Theorem 8.

Corollary 9. *Let $X \in M_{2n}$ be given. Then X has a ϕ_J polar decomposition if and only if (i) $X\phi_J(X)$ is similar to $\phi_J(X)X$ and (ii) $\operatorname{rank}([X\phi_J(X)]^p X)$ is even for every nonnegative integer p.*

Let $S \in \mathcal{S}_{2n}$ be given. Then there exists a nonsingular $Z \in M_{2n}$ such that $S = Z^TJZ$. Now, $X \in M_{2n}$ has a ϕ_S polar decomposition if and only if $Y \equiv ZXZ^{-1}$ has a ϕ_J polar decomposition [1]. Now, notice that (i) $\operatorname{rank}([X\phi_S(X)]^k) = \operatorname{rank}([Y\phi_J(Y)]^k)$ and that (ii) $\operatorname{rank}([X\phi_S(X)X]^k) = \operatorname{rank}([Y\phi_J(Y)]^kY)$.

Corollary 10. *Let $S \in \mathcal{S}_{2n}$ and $X \in M_{2n}$ be given. Then the following are equivalent:*

1. *X has a ϕ_S polar decomposition;*
2. $\operatorname{rank}([X\phi_S(X)]^i) = \operatorname{rank}([\phi_S(X)X]^i)$ *and* $\operatorname{rank}([X\phi_S(X)]^i X)$ *is even for every nonnegative integer i;*
3. *$X\phi_S(X)$ is similar to $\phi_S(X)X$ and* $\operatorname{rank}([X\phi_S(X)]^i X)$ *is even for every nonnegative integer i.*

Let $A, C \in M_{2n}$ and $B, D \in M_{2m}$ be given. Then $\operatorname{rank}(A \boxplus B) = \operatorname{rank}(A) + \operatorname{rank}(B)$ and $(A \boxplus B)(C \boxplus D) = AC \boxplus BD$. Moreover,

$$\phi_{J_{2n+2m}}(A \boxplus B) = \phi_{J_{2n}}(A) \boxplus \phi_{J_{2m}}(B).$$

Corollary 11. *Let $A \in M_{2n}$ and $B \in M_{2m}$ be given. If A and $A \boxplus B$ each have a ϕ_J polar decomposition, then so does B.*

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